# ON LINEAR AND NONLINEAR TRAJECTORY TRACKING CONTROL FOR NONHOLONOMIC INTEGRATOR

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*This paper presents two different kinds of trajectory tracking control strategies for the nonholonomic integrator known in literature as Brockett system. The first strategy presents a time-varying linear feedback control law and the second strategy is based on State Dependent Ricatti Equation (SDRE) method. Numerical simulation results indicated that both methods can be successfully used for control of the nonholomic integrator.*

***Keywords:*** *Brockett integrator, Hamilton – Jacobi – Bellman equation, SDRE method.*

## 1 Ntroduction

Over the past two decades the control problem of nonholonomic systems has became to attention. The main reason for this is the fact that there are large number of mechanical systems that have non-integrable constraints such as robot manipulators, mobile robots, wheeled vehicles, space and underwater robots [1], [2], and there are interesting problems in the scientific field such as control of molecular dynamics [3], nuclear magnetic resonance imaging and rotating electrical machinery [4].

Particularly, the autonomous mobile robots are system of great interest not only in academic studies but also in automotive industry, logistics machinery, aircraft industry and military applications. Mobile robot system models may contain nonholonomic constraints as, in example, of differential steering mobile robot. This system model is studied by [10], [7], [23] and the control problem of this system is challenging for traditional control methods. The two –wheeled differential steering mobile robot model can be reduced to the so called “nonholonomic integrator” introduced in control literature by Brockett in [5]. Sometimes it is referred to as Brockett's system or the Heisenberg system because it appears in quantum mechanics [6].

Several controllers were proposed for nonholonomic systems, most of them based on two main approaches which are posture stabilization and trajectory tracking. The problem of regulation control (or posture stabilization) is to stabilize a nonholonomic system at any given point in the state space; while the aim of trajectory tracking is to have the system following a reference trajectory.

The stabilization problem has received considerable attention in last decade. (See, for example, paper of Kolmanovsky and McClamroch [7]) The research efforts have been made to develop controllers based on either smooth dynamic feedback or nonsmooth feedback. Astolfi [8], [9], Canudas de Wit and Sordalen [10], Kolmanovsky and McClamroch [7], Morgansen and Brockett [11] and others used discontinuous approaches. In [12] Bloch and Drakunov [12] consider a sliding mode approach. Posture stabilization can also be achieved under time-varying continuous controls. (See, for example, Samson [13], Teel et al. [14], Pomet [15],).

The tracking problem has received less attention. In Walsh et al. [16] a locally exponentially stabilizing control was proposed. A dynamic feedback linearization technique for wheeled mobile robot was presented in Canudas de Wit and Sordalen [10]. Global tracking control laws were proposed in Jiang and Nijmeijer [17], Jiang [18] and Qu [19]. Bloch and Drakunov [11] used sliding mode control for trajectory tracking of Brockett integrator.

In this work two different kinds of trajectory tracking control strategies are presented for the nonholonomic integrator. The first strategy is a time-varying linear feedback control law [20] and the second strategy is based on State Dependent Ricatti Equation (SDRE) method [21].

This paper is organized as follows. In the section 2 the nonholonomic brocket integrator system is presented as well as the tracking control problem statement for this system. The optimal control problem formulation and the description of the optimal linear feedback control method is presented in section 3. The section 4 contains a description of the State Dependent Riccatti Equation control method and a suboptimal control problem formulation for the Brockett integrator. The section 5 is dedicated to the results obtained by numerical simulations of the controlled system for both control strategies. Finally, the concluding remarks follow in the section 6.

## 2 The control problem statement for the nonholonomic system

One of the examples of a simplest system with a nonholonomic constraint is the Brockett nonholonomic integrator introduced in [5]. This system has a following form:

$$\begin{aligned} \dot{x}_1 &= \bar{u}_1 \\ \dot{x}_2 &= \bar{u}_2 \\ \dot{x}_3 &= x_1\bar{u}_2 - x_2\bar{u}_1 \end{aligned} \qquad (1)$$

Where $x \in \Re^3$ is a state vector, $\dot{x} \in \Re^3$ is a time derivative of the state vector and $\bar{u} \in \Re^2$ is a control vector. The main goal is to realize the tracking control of this system, by minimizing its deviation from the reference trajectory:

$$\tilde{x} = \begin{bmatrix} \tilde{x}_1(t) \\ \tilde{x}_2(t) \\ \tilde{x}_3(t) \end{bmatrix} \qquad (2)$$

Then, the system (1) can be described by error coordinates $y(t) \in \Re^3$ expressing the difference between the system and a desired trajectory:

$$y = x \qquad (3)$$

Meanwhile the control vector $[\tilde{u}_1 \quad \tilde{u}_2]^T$ represents a feedforward control which maintains the system at the desired trajectory (2) and satisfies the following equation:

$$\begin{aligned} \dot{\tilde{x}}_1 &= \tilde{u}_1 \\ \dot{\tilde{x}}_2 &= \tilde{u}_2 \\ \dot{\tilde{x}}_3 &= \tilde{x}_1\tilde{u}_2 - \tilde{x}_2\tilde{u}_1 \end{aligned} \qquad (4)$$

The feedback control $u$ that realizes the tracking control of the system (1) to a trajectory (2) can be expressed as:

$$u = \bar{u} - \tilde{u} \qquad (5)$$

Therefore, error coordinate system is given by

$$\begin{aligned} \dot{y}_1 &= u_1 \\ \dot{y}_2 &= u_2 \\ \dot{y}_3 &= \tilde{u}_2 y_1 - \tilde{u}_1 y_2 + (\tilde{x}_1 + y_1)u_2 - (\tilde{x}_2 + y_2)u_1 \end{aligned} \qquad (6)$$

## 3 Optimal Linear State Feedback Control Problem Formulation

Considering the system:

$$\begin{aligned} \dot{y} &= A(t)y + h(y,u) + B(t)u \\ y(0) &= y_0 \end{aligned} \qquad (7)$$

where $y \in \Re^n$ is a state vector, $A(t) \in \Re^{n\times n}$ and $B(t) \in \Re^{n\times m}$ are bounded matrices, whose elements are time dependent, $u \in \Re^m$ is a control vector, and $h(y,u) \in \Re^n$ is a vector, whose elements are continuous nonlinear functions, with $h(0,0) = 0$.

Next, it is presented an important result, concerning a control law that guarantees stability for a nonlinear system and minimizes a nonquadratic performance functional.
*Theorem :* If there exist matrices *Q(t)* and *R(t)* , both positive definite being *Q* symmetric such that there will be a function in form:

$$l(y,u) = y^T Q.y - h^T(y,u)P.y - y^T.P.h(y,u) \quad , \quad (8)$$

that is positive definite, then the linear feedback control *u*:

$$u = -R^{-1}B^T P(t)y \quad (9)$$

is optimal to transfer the system (6) from the initial condition to a final state:

$$y(t_f) = 0, \quad (10)$$

minimizing the functional:

$$J = \int_0^{tf} [l(y,u) + u^T R(t) u \quad (11)$$

*P(t)* in eq. (9) is a positive definite symmetric matrix (for all $t \in [0, t_f]$) which is the solution of the matrix differential Ricatti equation :

$$\dot{P}(t) + P(t)A(t) + A(t)^T P(t) - P(t)B(t)R(t)^{-1}B(t)^T P(t) + Q(t) = 0 \quad , \quad (12)$$

satisfying the final condition:

$$P(t_f) = 0 \quad (13)$$

*Remark 2*: If $h(y,u) \equiv h(y)$ then the theorem above become the theorem formulated in [20] and their proofs are similar.
In the case of Brocket integrator, the system (6) can also be expressed in the form (7), with following matrices values:

$$A(t) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \tilde{u}_2 & -\tilde{u}_1 & 0 \end{bmatrix},$$

$$h(y,u) = \begin{bmatrix} 0 \\ 0 \\ y_1 u_2 - y_2 u_1 \end{bmatrix}, \quad (14)$$

$$B(t) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -\tilde{x}_2 & \tilde{x}_1 \end{bmatrix}.$$

## 4 SDRE Control Problem Formulation

SDRE method represents a systematic way of designing the nonlinear regulators [21], [22]. The explanation of the main idea of the method follows ahead.
Consider the general infinite-horizon, input-affine, autonomous, nonlinear regulator problem of the form:
Minimize:

$$J = \frac{1}{2}\int_0^{\infty} [y^T Q(y) y + u^T R(y) u] dt \quad (15)$$

with respect to the state *y* and control *u* subject to the nonlinear system constraints

$$\dot{y} = f(y) + g(y)u \quad (16)$$

where $y \in \Re^n$ is a state vector, $f(y)$, $g(y)$, $u \in \Re^n$ and matrices *Q(y)* and *R(y)* are positive definite for all *y*. We assume that $f(0) = 0$ and that $g(y) \neq 0$ in a neighborhood of the origin. The SDRE method requires following steps to obtain the suboptimal solution for the control problem (15)-(16) [22]:
i) Transformation of model (6) to a state dependent coefficient form so it become linear with state dependent coefficients:

$$\dot{y} = A(y)y + B(y)u, \quad (17)$$

where $f(y) = A(y)y$ e $B(y) = g(y)$.

ii) Solution of the state dependent Riccati equation

$$P(y)A(y) + A^T(y)P(y) - P(y)B(y)R^{-1}(y)B^T(y)P(y) + Q(y) = 0 \quad (18)$$

to obtain $P \geq 0$, where *P* is a function of *y*.

iii)Construction the nonlinear feedback controller

$$u = -R^{-1}(y)B^T(y)P(y)y \quad (19)$$

The system (6) placed in the state dependent coefficient form yields:

$$\dot{y} = A(y)y + B(y)u, \quad (20)$$

where $A(y) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \tilde{u}_2 & -\tilde{u}_1 & 0 \end{bmatrix}$ and

$$B(y) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -y_2 - \tilde{x}_2 & y_1 + \tilde{x}_1 \end{bmatrix}.$$

## 5 Numerical Simulation Results

Numerical simulations were made in order to demonstrate the results of both linear and nonlinear trajectory tracking, first, using the optimal time varying control formulation described in section 3 and then, using SDRE control formulation described in section 4. The numerical simulations were performed using the Runge-Kutta fourth order integration method with variable step to solve the system differential equations.

The desired trajectory was chosen as (21) for simulation of both control methods:

$$\tilde{x}(t) = \begin{bmatrix} \pi/4 \\ t \\ \pi t/4 \end{bmatrix} \quad (21)$$

Also the matrices *Q* and *R* were chosen constant for both methods:

$$Q = \begin{bmatrix} 100 & 0 & 0 \\ 0 & 100 & 0 \\ 0 & 0 & 100 \end{bmatrix}, R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad (22)$$

Then, two sets of initial conditions were chosen to demonstrate de performance of the control methods as follows the description. In first case, the chosen initial conditions are $x_0 = \begin{bmatrix} -\pi/2 \\ -3 \\ -1 \end{bmatrix}$ and figure 1 depicts the time evolution of the error coordinates for the optimal linear regulator method. The system matrices have following values:

$$A(t) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, h(y,u) = \begin{bmatrix} 0 \\ 0 \\ y_1u_2 - y_2u_1 \end{bmatrix}, \quad (23)$$

$$B(t) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -t & \pi/4 \end{bmatrix}.$$

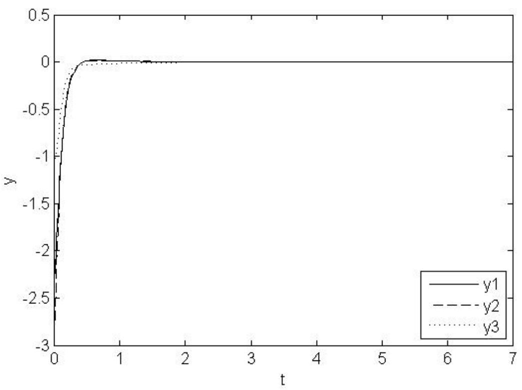


Figure 1. Error coordinates time evolution with initial conditions $x_0 = [-\pi/2 \quad -3 \quad -1]$ for Linear Feedback Method

The figure 2 shows the time evolution of the error system for the second presented method, the nonlinear regulator (SDRE control), where the system matrices have following values:

$$A(y) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \text{ and}$$

$$B(y) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -y_2 - t & y_1 + \pi/4 \end{bmatrix} \quad (24)$$

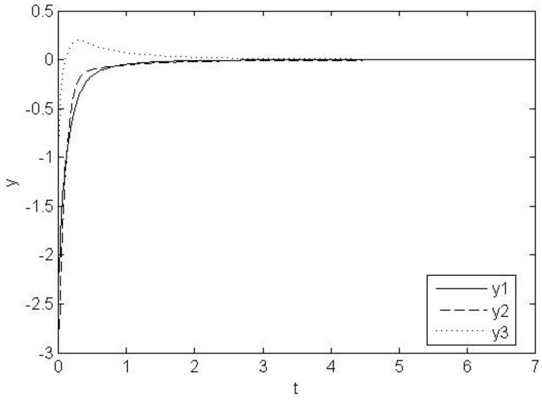


Figure 2. Error coordinates time evolution with initial conditions $x_0 = [-\pi/2 \quad -3 \quad -1]$ for SDRE Method

The second set of initial conditions was chosen as $x_0 = \begin{bmatrix} \pi/2 \\ -2 \\ 2 \end{bmatrix}$, then the figure 3 represents the time evolution of the error coordinates for the first control method (linear feedback ) with system matrices values as in (23).

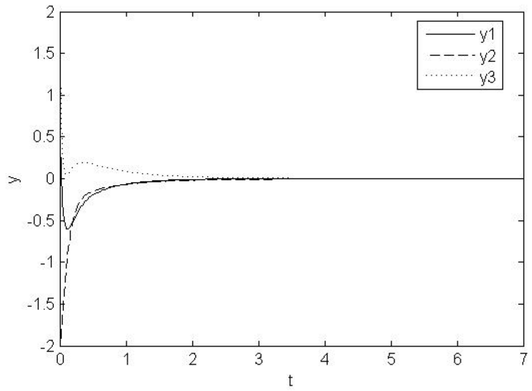


Figure 3 - Error coordinates time evolution with initial conditions $x_0 = [\pi/2 \quad -2 \quad 2]$ for Linear Feedback Method

Meanwhile, the figure 4 represents the time evolution of the error coordinates of the second control method (SDRE control), with system matrices values as in (24).

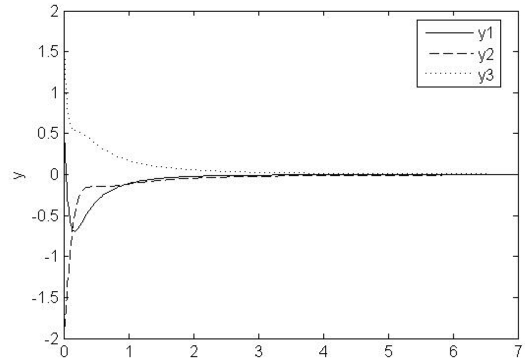


Figure 4 – Error coordinates time evolution with initial conditions $x_0 = [\pi/2 \quad -2 \quad 2]$ for SDRE Method

## 6 Conclusions

This paper has presented two tracking control strategies for the Brockett nonholonomic integrator. The first strategy is an optimal time-varying linear feedback control and the second strategy is based on State Dependent Ricatti Equation (SDRE), the suboptimal method. Numerical simulation results have demonstrated that both methods track the system to a chosen reference trajectory in the small amount of time, therefore, both can be successfully used for control of the nonholomic integrator. However, the SDRE method has an advantage of an online implementation.

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